

# Towards a Bell-Kochen-Specker theorem of identity

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In contrast to conventional, dynamical entanglement, in which particles with definite identity have uncertain properties, in so-called statistical entanglement, which arises between indistinguishable particles because of quantum symmetry rules, even particle identities are uncertain. The Bell and Kochen-Specker theorems imply that quantum properties either lack realism or possess it with a caveat of contextuality or nonlocality. In the matter of identity of multi-particle states of indistinguishable particles, these contrasting ontological attitudes are mirrored by the “bundle” vs. haecceity views. We offer some arguments aimed at importing the above theorems to the issue of identity in quantum theory, with the conclusion (under certain assumptions) that indistinguishable particles either lack individualism or possess a definite identity with a caveat of contextuality or nonlocality.

## I. INTRODUCTION

In classical physics, the notion of particle identity is not problematic. Particles are distinguishable individuals that can be specified via the Leibnizian “Principle of Indiscernibles” (PII). Any particle is “one of its kind” (can be defined in 2nd order logic, via quantification over properties). They possess haecceity (primitive thisness) and can be labelled, e.g., 1, 2, ...

In quantum mechanics, the formalism demands that multi-partitite states exist in which any two or more particles are physically indistinguishable *in principle*, implying a failure of PII. A system of indistinguishable bosons (resp., fermions) must be symmetric (resp., anti-symmetric) under pairwise exchange of particle labels, making these states entangled simply by virtue of the symmetry rules, rather than any interaction, as it is with conventional, dynamical entanglement. A subtlety here is that indistinguishable particles may have a definite relation among themselves: E.g., the two-boson state  $a_{\uparrow}^{\dagger}a_{\downarrow}^{\dagger}|\text{vac}\rangle$ , rendered in the first quantization language as the statistical entanglement state  $\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle)$ , corresponds to the statement “there exist two indistinguishable particles of *opposite* spin”. One might hope that this could be used to tell one particle from the other. Unfortunately this cannot be done because the relational property of oppositeness, corresponding to the proposition “has spin opposite to the other”, in the absence of definite spin of the “other”, does not lead to a monadic property that can distinguish the two particles.

In conclusion, one is led either to non-individualism, i.e., particles lacking definite individual identity, or to an *unphysical* haecceity. This situation parallels somewhat the implications of quantum mechanics for models purporting to explain the apparent randomness of outcomes of measurements corresponding to a physical property, such as position or spin along a given direction. The difference is that the uncertainty here pertains to particle *identity* rather than to a *property* that a particle of given identity possesses.

As applied to possessed properties, this uncertainty was considered by Einstein, Podolsky and Rosen [1] as indicative of the incompleteness of quantum mechanics (QM). They expressed hope for the possibility of completing QM with variables that possessed *realism*, i.e., definite values of properties prior to measurement, which are revealed, and not created during measurement. The definitive answer to this problem in standard QM are the Bell [2–4] and Kochen-Specker (KS) [5, 6] theorems, which assert that in certain situations, quantum mechanical properties either lack realism or are realistic (governed by some hidden variables theory) with qualifications of nonlocality or contextuality, respectively. We will generally refer to the import of these theorems as the uncertainty of possessed properties of quantum systems. For reasons we clarify elsewhere, both theorems will be regarded here as facets of the same quantum phenomenon. It may be said that this uncertainty of identity, which has recently received a lot of attention [7–11], heightens the quantum ‘mystery’ already evident in the uncertainty of the possessed property.

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## II. CONTEXTUAL REALISM OF THE POSSESSED PROPERTY

The contradiction obtained by the Bell and KS theorem has its origin in the fact that observables corresponding to properties are not required to commute in QM, rendering it impossible to embed the algebra of these observables in a commutative algebra, taken to represent the classical structure of the putative hidden variables. An illustrative example is discussed below.

The GHZ nonlocality proof [12] considers a three-qubit system in the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle), \quad (1)$$

on which the mutually commuting three-qubit measurements  $XYX$  or  $YXY$  or  $YYX$  or  $XXX$  may be performed, where  $X$  and  $Y$  refer to the Pauli operators along those directions, and a tensor product is assumed between any two operators. The state  $|\Psi\rangle$  is a  $+1$ -eigenstate of the first three operators, and a  $-1$ -eigenstate of the last. A realist model to explain quantum indeterminacy of this system requires that there exist definite values of the  $X_j$ 's and  $Y_k$ 's, where  $j, k = x, y, z$ , indicate the position index, and particles  $a, b$  and  $c$  are localized at  $x, y$  and  $z$ , respectively.

That no such assignment is possible is seen from the following table:

$$\begin{array}{llll} X_x & Y_y & Y_z & \rightarrow +1 \\ Y_x & X_y & Y_z & \rightarrow +1 \\ Y_x & Y_y & X_z & \rightarrow +1 \\ X_x & X_y & X_z & \rightarrow -1 \end{array} \quad (2)$$

Any realist assignment of  $\pm 1$  to the  $X_j$ 's and  $Y_k$ 's will yield a product of  $+1$  along the first three columns because there are two copies of  $X$  or  $Y$  along each column. The product of these products is  $+1$ , which contradicts the  $-1$  obtained above.

However a nonlocal-realistic explanation is possible: for example, let all  $X$ 's and  $Y$ 's  $= 1$ , except  $X_x$ , which will be  $+1$  in first row and  $-1$  in the last.

## III. ALTERNATE VIEW: CONTEXTUAL HAECCEITY

We wish to use a Bell-Kochen-Specker-like argument to clarify the sense in which particles have uncertain identities. We require a multi-partite system, rather than a single-particle quantum system for which the Kochen-Specker theorem is usually proved. It turns out that the GHZ proof of nonlocality [12] for 3 particles, suits our purpose. One difference worth noting is that properties have a corresponding observable that can make them definite under measurement, whereas no such *identity observable* exists within the 'language' of quantum mechanics [13]. Another point to note is that multi-partite interference cannot be thought of as interference of individual particles caused by indistinguishability [14]. Instead, they are interferences of two-particle or multi-particle amplitudes. For example, in the well-known Hong-Ou-Mandel interferometer, the interference happens between the 'both-transmit' and 'both-reflect' amplitudes [15].

The state  $|\Psi\rangle$  in Eq. (1) can be interpreted as the following 3-boson state in a quantum optical situation:

$$\begin{aligned} |\Psi\rangle &\equiv (a_{0,z}^\dagger a_{0,y}^\dagger a_{0,z}^\dagger - a_{1,z}^\dagger a_{1,y}^\dagger a_{1,x}^\dagger) |\text{vac}\rangle \\ &= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle)_{abc} \otimes \frac{1}{\sqrt{6}} \Pi_{[x,y,z]} |\psi_x\rangle |\psi_y\rangle |\psi_z\rangle, \end{aligned} \quad (3)$$

which represents bosonic particles created at positions  $x, y$  and  $z$  in the GHZ state, and  $\Pi$  represents a permutation of the spatial wave functions over all formal particle labels. The state remains invariant under label exchanges  $a \leftrightarrow b$ ,  $a \leftrightarrow c$ ,  $b \leftrightarrow c$ . Any spin measurement  $O_x$  on the first position should be represented as a symmetrization over all three formal particle labels as  $O_x \otimes I_b \otimes I_c + I_a \otimes O_x \otimes I_c + I_a \otimes I_b \otimes O_x$ . Just as quantum superposition expresses that a particle's property lacks definiteness, a state like Eq. (3) says that in the 3-particle state, single particle identities are not definite, in fact, they are *maximally* indefinite.

Suppose the 3 particles have definite individual identity, bound with a definite spin value but there is freedom to nonlocally (or contextually) influence the particle identity associated with a given position. That the data in Table 2 can be explained follows from the assignment table

$$\begin{bmatrix} X_x & Y_y & Y_z \\ Y_x & X_y & Y_z \\ Y_x & Y_y & X_z \\ X_x & X_y & X_z \end{bmatrix} \leftarrow \begin{bmatrix} b & a & c \\ a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}, \quad (4)$$

with spin value assignments being  $X_b = Y_a = Y_b = -1$  and  $X_a = X_c = Y_c = +1$ . This says that the identity of the particles at  $x$  and  $y$  are  $b$  and  $a$ , respectively, in the situation corresponding to the top row, but the opposite in the situation corresponding to the last row.

#### IV. CONCLUSION AND DISCUSSION

The uncertainty of individual identity in quantum mechanics because of the symmetry rules, among one of the earliest known non-classical features, is seen in this light to make quantum mechanics even weirder in a world already familiar with entanglement. While identity uncertainty is, in this sense, similar to uncertainty of the possessed property, the former comes with features absent in the latter, and the implications of these would have to be studied. As noted, there is no identity observable, let alone its maximally non-commuting conjugate, so that the role of non-commutativity, central to the Bell and Kochen-Specker theorems, is borrowed from the property algebra. As a result, the contradiction obtained here requires an identity being bound to a definite value of the property. If this binding were relaxed, and particle identities bound with the positions, then the standard nonlocal realistic explanation of the GHZ contradiction, as given in Section II, is also applicable. Of interest would be a contradiction, if it exists, between quantum mechanics and a local/non-contextual assignment of identity, even allowing for nonlocal/contextual properties, perhaps using HOM interferometers, quantum erasers, etc.

#### Acknowledgments

We are thankful to Ms. Akshata H. Shenoy for helpful discussions.

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